$\boxed{8}$ Revisiting $D(Bun_G)$

<u>conjecture</u>

$$D(Bun_G) \simeq IC_{\mathcal{N}^{\circ}_{\check{G}}}(Flat_{\check{G}})$$

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1) Conjecture for $G = \mathbb{G}_m$

It is known

$Bun_{\mathbb{G}_m}$?

on $C = \mathbb{P}^1$
$\mathcal{L} = \mathcal{O}(n)$
$n \in \mathbb{Z}$

Recall <u>$Pic(C)$</u> Picard scheme

<u>$Pic^{\circ}(C)$</u> deg 0 part

$\cong Jac\, C$

want to write

$$Bun_{\mathbb{G}_m} \simeq Pic\, C \underset{\substack{\uparrow \\ \text{non-canonical}}}{\simeq} Jac\, C \times \mathbb{Z} \times B\mathbb{G}_m$$

$G = \mathbb{G}_m \longleftrightarrow \mathbb{G}_m = \check{G}$
$G = Gl_n \longleftrightarrow Gl_n = \check{G}$

$$Flat_{\mathbb{G}_m} \underset{\substack{\uparrow \\ \text{non-canonical}}}{\simeq} \underline{Flat_1} \times B\mathbb{G}_m \times Spec\, k[\eta]$$

$\deg \eta = -1$

$$\mathbb{T}\, Flat_{\mathbb{G}_m} \simeq (\Omega^{\bullet}, d_{dR}) \quad \text{in smooth cat.}$$

$$= (\overset{-1}{\Omega^0} \to \overset{0}{\Omega^1} \to \overset{1}{\Omega^2} \cdots)$$

$\mathcal{N} \subset \mathfrak{g}^*/G$

$G = T$

$\Rightarrow \mathcal{N} = 0$

$\Rightarrow IC_{\mathcal{N}} = QC$

Claim

$D(Jac) \simeq QC(Flat_1)$

$D(\mathbb{Z}) \simeq QC(\mathbb{B}G_m)$

$D(\mathbb{B}G_m) \simeq QC(\operatorname{Spec} k[\eta])$

① $T^*Jac \simeq Jac \times H^0(C, \Omega_C)$

$$\begin{tikzcd} QC(T^*Jac)_{FM} \arrow[r, phantom, "\cong"] & QC(T^*Jac) \arrow[d, squiggly, "\text{deformation}"] \\ D(Jac) \arrow[r, phantom, "\underset{FMLR}{\cong}"] & QC(Flat_1) \end{tikzcd}$$

② ~~BG~~ $D(\mathbb{Z}) = \mathbb{Z}$-graded Vect

Claim $QC(\mathbb{B}G) \simeq \operatorname{Rep} G$

$\rightsquigarrow QC(\mathbb{B}G_m) \simeq \operatorname{Rep} G_m$

③ $\underline{rhs} = k[\eta]$-mod

Goal understand $D(\mathbb{B}G_m)$

Recall $\mathcal{X}$ prestack

$$D^{l}(\mathcal{X}) = QC(\mathcal{X}_{dR})$$

where $\mathcal{X}_{dR}(S) := \mathcal{X}(S^{red})$

category of left D-modules

Defn $D^{r}(\mathcal{X}) := IC(\mathcal{X}_{dR})$

// right D-modules

$$D^{l}(\mathcal{X}) \xrightarrow{\gamma_{x}} D^{r}(\mathcal{X}) \qquad \text{where } \omega_{\mathcal{X}} = p_{\mathcal{X}}^{!} k$$

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \omega_{x} \qquad \text{for } p_{\mathcal{X}} : \mathcal{X} \to pt$$

Six functors formalism

$f: X \to Y$

(f^{*}, f_{*}) $(f_{!}, f^{!})$ adjoint pairs $\otimes$, Hom

pullback

$\int$ along fiber

$\int$ of cpt. supp of fiber

Verdier duality $\mathbb{D}$

$f^{*} = \mathbb{D} f^{!} \mathbb{D}$

- If f is proper then $f_{!} = f_{*}$
- If f is smooth ~~then~~ of relative real dim d $f^{!} = f^{*}[d]$

Review of Koszul duality

$$\begin{tikzcd} Z_\Gamma \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] \\ pt \arrow[r, "\{0\}"] & \mathbb{A}^m \end{tikzcd} \rightsquigarrow \begin{tikzcd} \operatorname{spec} k[\eta] \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$|\eta| = -1$

$\left(k \otimes^L_{k[x]} k = k[\eta]\right)$

Prop $QC(\operatorname{Spec} k[\eta]) = k[\eta]\text{-mod} \xrightarrow{KD} \overset{\text{Vect}}{k[\xi]}\text{-mod}$

$$M \longrightarrow \underline{Hom}_{k[\eta]}(k, M)$$

① $k \longrightarrow \underline{Hom}_{k[\eta]}(k,k) = k[\xi]$

conclude $KD : k[\eta]\text{-mod} \to k[\xi]\text{-mod}$

~~Hom~~ $V = k\langle x \rangle$ $\quad W = V[1]$

$\uparrow$
$x_1, \dots x_m$

$$\underline{Hom}_{\operatorname{Sym} W}(k,k)$$

$$k \simeq \operatorname{Sym}(W[1] \oplus W) \quad d = id$$

$\overset{-1}{\mathbb{C}^2} \xrightarrow{d=id} \overset{0}{\mathbb{C}^2}$
$\Rightarrow$ trivial

$[\cdots \to \operatorname{Sym} W \otimes \operatorname{Sym}^2 W[1] \to \operatorname{Sym} W \otimes W[1] \to \operatorname{Sym} W \to k]$ Koszul resolution

$$\underline{Hom}_{\operatorname{Sym} W}(\operatorname{Sym}(W[1] \oplus W), k)$$
$$= \underline{Hom}(\operatorname{Sym}(W[1]), k) = \operatorname{Sym}(W^*[-1])$$
$$= k(\xi)$$

② KD is fully faithful on k

pf: $Hom_{k[\eta]}(k,k] = k[\xi] = Hom_{k[\xi]}(k[\xi], k[\xi])$

conclude KD is fully faithful on $\operatorname{Perf}(\operatorname{Spec} k[\eta])$

$\text{Perf}(\text{Spec } k[\xi])$
is f.g. over k
because $k[\eta]$ is Artinian

$$\begin{tikzcd}
\text{f.g. over } k \arrow[r, "\sim"] & \text{f.g. over } k[\xi] \\
\text{Perf}(\text{Spec } k[\eta]) \arrow[u, phantom, "\cup"] \arrow[ur, hook] &
\end{tikzcd}$$

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③ $\text{Coh}(\text{Spec } k[\eta]) \xrightarrow{\sim} k[\xi]\text{-mod}^{f.g.}$

Summary To understand $QC(\text{Spec } k[\eta])$
we find a generator k and show
$QC(\quad) \sim \text{Hom}(k,k)\text{-mod}$

Rmk (noncommutative geometry)

$$\begin{tikzcd}
\text{Fuk} \;\big|\; \text{Coh} & \text{Try} & \text{Coh}(X) \arrow[d, phantom, "\shortparallel" near start] \simeq A\text{-mod} \\
X \text{ symp} \;\big|\; X \text{ CY-manifold} & \mathcal{O}_X\text{-mod} & \text{Ext}(g_1 \oplus \cdots \oplus g_n) \arrow[u] \\
& & \text{self-ext} \arrow[u]
\end{tikzcd}$$

To understand $D(BG_m)$ what should we do?
need to find a nontrivial obj
Claim! local systems on BG_m are all trivial
$$\pi_1(BG) = \pi_0(G) \Rightarrow \pi_1(BG_m) \xrightarrow[\text{trivial}]{} GL_n$$
$$\pi_0(G_m) = \{1\}$$

$$\cdots \mathbb{G}\times\mathbb{G}\times\mathbb{G} \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \mathbb{G}\times\mathbb{G} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbb{G} \rightrightarrows B\mathbb{G}$$

$\omega_{B\mathbb{G}_m} = p^! k \quad p : B\mathbb{G}_m \to pt$

$k \leftrightarrow \omega$

$\mathrm{Hom}_{D(B\mathbb{G}_m)}(\omega_{B\mathbb{G}_m}, \omega_{B\mathbb{G}_m}) = dR(B\mathbb{G}_m)$

$\cong k[\xi]$

$|\xi| = 2$

$D(B\mathbb{G}_m) \to k[\xi]\text{-mod}$

$\omega \longrightarrow \mathrm{Hom}(\omega, \omega)$

$M \mapsto \underline{\mathrm{Hom}}_{k[\eta]}(k, M)$

equiv over M f.g.

$0 \to \omega[1] \to g \to \omega \to$

$QC(\mathrm{Spec}\ k[\eta])$

cpt. generator $k[\eta] \underset{\text{vector space}}{\cong} k \oplus k[1]$

$\Big\Vert$ - - - - $\updownarrow$ $\Big($ $\mathrm{Ext}^1(k, k[1]) = \mathrm{Ext}^2(k,k)$

$\nearrow$ $D(B\mathbb{G}_m)$ $g \cong \bar{\omega} \oplus \bar{\omega}[1]$ - -

$\mathrm{Hom}(g,-)$ cpt. generator

$\xi \in \mathrm{Ext}^2(\omega, \omega)$

$= \mathrm{Ext}(\omega, \omega[1])$

$\mathrm{Hom}_{D(B\mathbb{G}_m)}(g, g)\text{-mod} \cong QC(\mathrm{Spec}\ k[\eta])$

$\Vert$

$k[\eta]$

$QC(\mathrm{Spec}\ k[\eta])$	$D(B\mathbb{G}_m)$
k	ω
$k[\eta]$	g
$\mathrm{Hom}_{k[\eta]}(k[\eta], k[\eta]) = k[\eta]$	$\mathrm{Hom}_{D(B\mathbb{G}_m)}(g, g) = k[\eta]$
$\mathrm{Hom}_{k[\eta]}(k, k) = k[\xi]$	$\mathrm{Hom}_{D(B\mathbb{G}_m)}(\omega, \omega) = k[\xi]$

$$\mathrm{Bun}_G = G(k(C)) \backslash G(\mathbb{A}) / G(\mathcal{O})$$

G semisimple

G-bundle on C can be understood as triv. G-bundles P_1 on D_x + triv G-bundle P_2 on $C \setminus \{x\}$ and identification

$$P_1|_{D_x^\times} = P_2|_{D_x^\times}$$

$$\mathrm{Gr}_G = G(K)/G(\mathcal{O})$$ affine Grassmannian

$K = \mathbb{C}((t))$

$\mathcal{O} = \mathbb{C}[[t]]$

moduli space of G-blls P on D w/ trivializations $P|_{D^\times} \cong P^0|_{D^\times}$

$G = GL_n$

$Gr_G \simeq \{\text{lattices in } K^n\}$

$L \subset K^n$

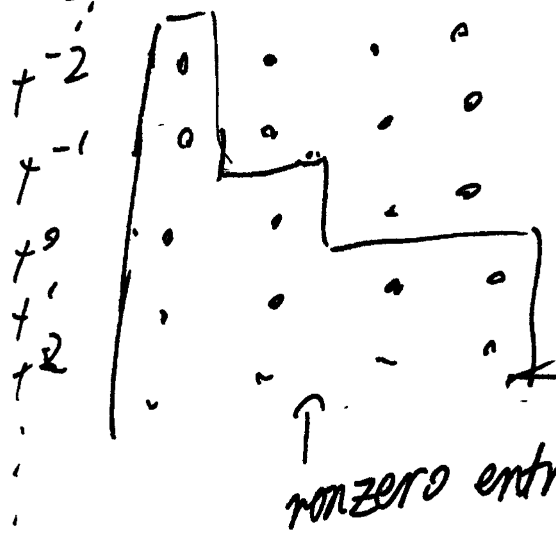
$t \cdot L \subset L$

pf) $O^n \subset K^n$ lattice

$G(K)$ transitive action $\quad t^N O^N \subset L \subset t^{-N} O$

$G(O)$ stabilizer $\quad \forall N \gg 0$

Picture $n=4$:



dots = basis of lattice

lattice

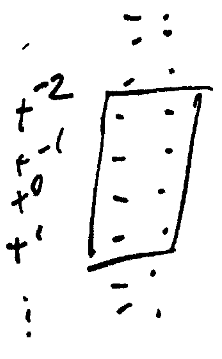
nonzero entries

(can be made into descending stairs post. permutation)

$Gr_{GL_n} = \bigcup_{k \in \mathbb{Z}_{\geq 0}} Gr_k$

where Gr_k one can take $N = k$ ~~in the tightest way~~

$n=2 \quad k=2$



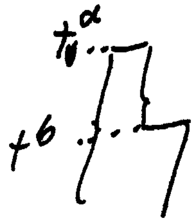
$t^{-2} O^2 / t^2 O^2$

$= \mathbb{C}^8$

$t = \left(\begin{array}{c|c} \begin{smallmatrix}0&1&&\\&0&1&\\&&0&1\\&&&0\end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix}0&1&&\\&0&1&\\&&0&1\\&&&0\end{smallmatrix}\end{array}\right)$

Gr_G ind-proj
(colim of proj. var)

$Gr_k = \coprod_{-k\le a,b\le k} Gr_{a,b}$



t[illegible] $L \subset t^a\mathcal{O}^2$

$t^{b-1}\mathcal{O}^2 \not\subset L \not\subset t^{a+1}\mathcal{O}^2$

<u>Prop</u> $\overline{Gr_{a,b}} = \coprod_{0\le i\le \frac{1}{2}(b-a)} Gr_{a+i,b-i}$

pf) $\alpha_{a,b} = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} \in GL_2(K)$

find a sequence

s.t. $\underset{\substack{\uparrow \\ Gr_{a,b}}}{\{\alpha_k\}_{k=1}^{\infty}} \to \alpha_{a+i,b-i}$

$$\alpha_k = \left\langle \overset{\substack{t^b\mathcal{O}^2 \\ \cap}}{\underset{\substack{\cap \\ t^a\mathcal{O}^2}}{t^{a+i}e_1}}, \frac{1}{k}t^a + t^{b-i}e_2 \right\rangle$$

consider $\wedge^2 : Gr_{GL_2} \to Gr_{GL_1} = \mathbb{Z}$

$\alpha_{a,b} \to t^{a+b}\mathcal{O}$

$\Rightarrow \coprod_{a+b=j} Gr_{a,b}$ is closed ∎

Gr_{GL_2} connected components?

$\coprod_{i\geq 0} Gr_{a-i,\,a+i}$ $\qquad$ $\coprod_{i\geq 0} Gr_{a-i,\,a+1+i}$ $\qquad a \in \mathbb{Z}$

$\pi_0(Gr_{GL_2}) = \mathbb{Z}$

Gr_{SL_2} cpt. $= \coprod_{i\geq 0} Gr_{-i,i}$

$\pi_0(Gr_{SL_2}) = \{1\}$ $\qquad\qquad \pi_0 = \mathbb{Z}_2$

$Gr_{PGL_2} = \coprod_{i\geq 0} Gr_{0,2i}\,,\ \coprod_{i\geq 0} Gr_{0,2i+1}$

$\pi_0(Gr_G) = \pi_1(G) \underset{\uparrow}{=} \mathbb{Z}(\check{G})$

G simple

$G = SL_2 \Rightarrow 1 = \pi_1(SL_2) \cong Z(PGL_2)$

$G = PGL_2 \quad \pi_1(PGL_2) = C_2 \cong Z(SL_2)$

$Gr_G \simeq \Omega K$

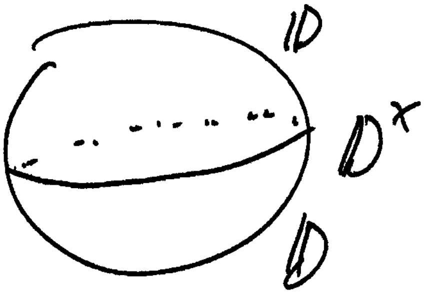
$K \subset G$ cpt gp.

$GL_2 \rightsquigarrow Gr_{a,b}$

G reductive gp $\rightsquigarrow \lambda \in \Lambda_G^+$ dominant coweights

$Gr_G = \coprod_{\lambda \in \Lambda_G^+} Gr_\lambda$ schubert varieties

$\nwarrow G(\mathcal{O})$ - equivariant orbit



$$\mathrm{Bun}_G(\mathbb{P}^1) \simeq G(k[t^{-1}]) \backslash G(k((t))) / G(k[[t]])$$

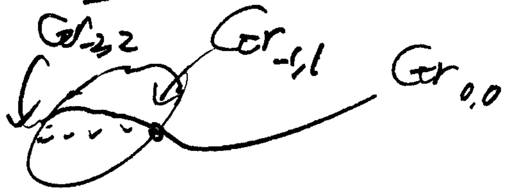
$$G = GL_n$$

Thm (Birkhoff, Grothendieck) any vector bundle of rank n over $\mathbb{P}^1$ is iso to $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$

connected components is parameterized by $\sum k_i$

$\mathcal{O} \oplus \cdots \oplus \mathcal{O} \in$ cmpt. w/ $\sum k_i = 0$

Gr_{GL_2}



$\cdots$ $\mathcal{O}(-2) \oplus \mathcal{O}(2)$ $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ $\mathcal{O} \oplus \mathcal{O}$ in $\mathrm{Bun}_{GL_2} \mathbb{P}^1$

$\mathcal{O} \oplus \mathcal{O}$ is open dense